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VARIATIONAL PRINCIPLES FOR ELASTIC PLATES WITH RELAXED CONTINUITY REQUIREMENTS*

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Abstract—The classical variational principles of the theory of elastic plates impose stringent continuity conditions on the moment and deflection fields that are admitted for comparison with the natural fields. Since these continuity conditions are difficult to fulfil in a finite-element approach, the classical variational principles cannot be used to derive the basic equations of typical finite element methods. Modified variational principles are established that impose less exacting continuity conditions on the relevant fields. These principles are classified according to the number of independent fields that are involved. The application of the principles to typical finite-element analyses is indicated.

1. INTRODUCTION

THE classical variational principles of the theory of elastic plates impose continuity requirements on the fields of deflection or bending and twisting moments that impair their usefulness in the derivation of valid finite difference or finite element approximations. The principle of minimum potential energy, for instance, requires the use of continuous deflection fields that have continuous first and piecewise continuous second derivatives. This requirement precludes the use of the most convenient type of deflection field, which transforms the originally plane median surface of the plate into a polyhedron with triangular facets. The present paper is concerned with variational principles for elastic plates that impose less stringent continuity conditions on the deflection and moment fields. Whereas some principles of this kind have been introduced in *ad hoc* fashion for finite element formulations (see, for instance, [1]), the present systematic derivation of the relevant principles is believed to be new. Analogous principles for the three-dimensional continuum were discussed in an earlier paper [2].

2. FIELD EQUATIONS, BOUNDARY AND DISCONTINUITY CONDITIONS

The plate is treated as a two-dimensional elastic continuum that occupies a simply connected region A with boundary s in the plane of the rectangular Cartesian coordinates x_i, $(i = 1, 2)$. Throughout A, the intensity p of the transverse load is given as a function of position.

The complete solution of a plate problem involves three fields in addition to the given load field p: the moment field m_{ij}, the deflection field w, and the curvature field $\varkappa_{ij}$. Across a line element in the x_2-direction, the specific† bending moment m_{11}, the specific twisting

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† Depending on the context in which it is used, the term "specific" means "per unit length" or "per unit area."

moment $m_{12} = m_{21}$, and the specific shear force* $\partial_i m_{i1}$ are transmitted. Equilibrium of the plate element requires that†

$$\partial_{ij} m_{ij} + p = 0. \tag{1}$$

The curvature tensor $\varkappa_{ij}$ is defined by

$$\varkappa_{ij} = -\partial_{ij} w_{ij}; \tag{2}$$

it is related to the moment tensor by

$$m_{ij} = C_{ijkl}\varkappa_{kl}, \qquad \varkappa_{ij} = \bar{C}_{ijkl} m_{kl}, \tag{3}$$

where

$$\left.\begin{aligned} C_{ijkl} &= D[(1-\nu)\delta_{ik}\delta_{jl} + \nu\delta_{ij}\delta_{kl}], \\ \bar{C}_{ijkl} &= \frac{1}{(1-\nu^2)D}[(1+\nu)\delta_{ik}\delta_{jl} - \nu\delta_{ij}\delta_{kl}]. \end{aligned}\right\} \tag{4}$$

In (4), δ_{ij} is the Kronecker delta, which has the values 1 or 0 according to whether the subscripts i and j have or do not have the same numerical value; ν is Poisson's ratio, and

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{5}$$

the plate stiffness, E being Young's modulus and h the thickness of the plate, which need not be constant. Equations (1) through (3) will be referred to as the *field equations* of the plate problem.

In the x_1, x_2-plane, consider an arbitrary, oriented line element $\mathrm{d}s$ with the unit tangential vector t_i and the unit normal vector n_i obtained from t_i by a clockwise rotation of 90°. The bending moment M, the twisting moment T, the shear force Q, and the effective shear force V transmitted across $\mathrm{d}s$ are

$$\begin{aligned} M &= m_{ij} n_i n_j, \qquad & T &= m_{ij} n_i t_j, \\ Q &= n_j \partial_i m_{ij}, \qquad & V &= Q + \mathrm{d}T/\mathrm{d}s. \end{aligned} \tag{6}$$

As the plate bends, it assumes the slope

$$\theta = n_i \partial_i w \tag{7}$$

in the direction of n_i.

The boundary conditions considered in the following require that either M or θ and either V or w are prescribed for any boundary element $\mathrm{d}s$. The twisting moment T and the shear force Q do not appear explicitly in these boundary conditions. Superimposed bars will be used to indicate the given boundary values $\bar{M}$ or $\bar{\theta}$ and $\bar{V}$ or $\bar{w}$.

Not all of the fields m_{ij}, w, $\varkappa_{ij}$ need to appear in the statement of a variational principle. Moreover, fields that do appear need not be required to satisfy the appropriate boundary conditions. A field will be called constrained or free according to whether it is or is not required to satisfy these boundary conditions.

* The symbol ∂_i indicates differentiation with respect to x_i, and repeated letter subscripts indicate summation over the range 1, 2. Thus, $\partial_i m_{i1} = \partial_1 m_{11} + m_{21}$.

† The symbol ∂_{ij} is an abbreviation for $\partial_i \partial_j$.

Each of the field equations (1) through (3) links two fields, which will be said to "correspond" to each other. For example, a moment field m_{ij} and "the corresponding curvature field" $\varkappa_{ij}$ must satisfy (3).

The terminology introduced above facilitates the characterization of the fields involved in a variational principle. For example, the classical principle of minimum potential energy for elastic plates involves a constrained deflection field and the corresponding curvature field.

The bending moment, deflection, and curvature fields considered in the following are supposed to be divided into contiguous domains of regularity, in each of which the field quantities have the continuity and differentiability properties assumed in the fundamental equations of the theory of elastic plates. Adjacent domains of regularity are separated by a line of discontinuity. If this line of discontinuity is regarded as part of the boundary of one of the adjacent domains of regularity, the quantities M, T, Q, V, and θ for this domain at a given point P of the line of singularity are again defined by (6) and (7), where n_i now is the unit vector along the exterior normal of the considered domain at P, and t_i is obtained from n_i by a counterclockwise rotation by 90°. Since, however, the element of a line of discontinuity will be denoted by $\mathrm{d}\sigma$ to distinguish it from the element $\mathrm{d}s$ of the plate boundary, $\mathrm{d}T/\mathrm{d}s$ in the last expression (6) should be replaced by $\mathrm{d}T/\mathrm{d}\sigma$.

To specify a discontinuity at a point P of a line of discontinuity σ, arbitrarily label the two sides of σ as positive and negative, and use the subscripts + and − to refer to these sides. As one proceeds normally to σ from the negative to the positive side, it follows from the definitions (6) and (7) that the vanishing of the *jumps*

$$\begin{aligned} &\langle w\rangle = w_+ - w_-, \qquad \langle M\rangle = M_+ - M_-, \qquad \langle T\rangle = T_+ - T_-, \\ &\langle \theta\rangle = -(\theta_+ + \theta_-), \qquad \langle Q\rangle = -(Q_+ + Q_-), \qquad \langle V\rangle = -(V_+ - V_-) \end{aligned} \tag{8}$$

implies that the deflection and moment fields are continuous and have continuous first derivatives.

The discontinuities considered in the following will be described in terms of four of the jumps in (8), and it will be assumed that

$$\langle M\rangle\langle\theta\rangle = 0 \quad \text{and} \quad \langle V\rangle\langle w\rangle = 0 \tag{9}$$

along each line of discontinuity. Notations such as $\int_M M\theta\, d\sigma$ will be used to indicate that the integral of $M\theta$ along one side of σ should only be extended over those parts where $\langle M\rangle \neq 0$ and hence $\langle\theta\rangle = 0$. Similarly, the notation $\int_M M\theta\, \mathrm{d}s$ will be used to indicate that the integral should only be extended over those parts of the plate boundary on which M is prescribed.

It will be convenient to denote by $\int\varphi\, \mathrm{d}A$ the sum of the integrals of the field quantity φ over *all* domains of regularity, and by $\int\varphi\, \mathrm{d}s$ or $\int\varphi\, \mathrm{d}\sigma$ the sums of the integrals of φ over those parts of the boundaries of these domains that respectively fall on the plate boundary or on lines of discontinuity. Note that each element occurs twice in $\int\varphi\, \mathrm{d}\sigma$ because it is on the boundary of two adjacent domains.

For free and independent fields m_{ij} and w, one then has the identity

$$\begin{aligned} \int m_{ij}\partial_{ij}w\, \mathrm{d}A = \int \partial_{ij}m_{ij}w\, \mathrm{d}A + \int M\theta\, \mathrm{d}s - \int Vw\, \mathrm{d}s \\ + \int M\theta\, \mathrm{d}\sigma - \int Vw\, \mathrm{d}\sigma + [Tw]. \end{aligned} \tag{10}$$

The last term in (10) takes account of the fact that the boundary of the typical domain of regularity may consist of a finite number of sides with continuously turning tangent that meet at corners. The contribution of a side to the expression $[Tw]$ then is obtained by subtracting the value of Tw for the considered domain at the origin of this side from its value at the terminal.

3. THREE INDEPENDENT FIELDS

A variational principle involving free and independent fields of moment, deflection, and curvature states that the natural fields are characterized by the variational equation $\delta I_1 = 0$, where

$$
\begin{aligned}
I_1(m, w, \varkappa) = & \int \{\tfrac{1}{2}C_{ijkl}\varkappa_{ij}\varkappa_{kl} - m_{ij}(\partial_{ij}w + \varkappa_{ij}) - pw\}\, \mathrm{d}A + \int_M \bar{M}\theta\, \mathrm{d}s - \int_V \bar{V}w\, \mathrm{d}s \\
& - \int_\theta M(\bar{\theta} - \theta)\, \mathrm{d}s + \int_w V(\bar{w} - w)\, \mathrm{d}s + \int_\theta M\theta\, \mathrm{d}\sigma - \int_w Vw\, \mathrm{d}\sigma.
\end{aligned} \tag{11}
$$

Indeed, one finds

$$
\begin{aligned}
\delta I_1 = & \int (C_{ijkl}\varkappa_{kl} - m_{ij})\delta\varkappa_{ij}\, \mathrm{d}A - && \text{(Hooke's law)} \\
& - \int (\partial_{ij}w + \varkappa_{ij})\, \delta m_{ij}\, \mathrm{d}A - && \text{(definition of curvature)} \\
& - \int (\partial_{ij}m_{ij} + p)\, \delta w\, \mathrm{d}A + && \text{(equilibrium)} \\
& + \int_M (\bar{M} - M)\, \delta\theta\, \mathrm{d}s - \int_\theta (\bar{\theta} - \theta)\, \delta M\, \mathrm{d}s - && \text{(boundary conditions on } M \text{ and } \theta) \\
& - \int_V (\bar{V} - V)\, \delta w\, \mathrm{d}s + \int_w (\bar{w} - w)\, \delta V\, \mathrm{d}s - && \text{(boundary conditions on } V \text{ and } w) \\
& - \int_M M\, \delta\theta\, \mathrm{d}\sigma + \int_\theta \theta\, \delta M\, \mathrm{d}\sigma + && \text{(continuity of } M \text{ and } \theta) \\
& + \int_V V\, \delta w\, \mathrm{d}\sigma - \int_w w\, \delta V\, \mathrm{d}\sigma - && \text{(continuity of } V \text{ and } w) \\
& - [T\, \delta w]. && \text{(equilibrium of corner forces)}
\end{aligned} \tag{12}
$$

In (12), the consequences of the arbitrariness of the variations in each integral are displayed on the right, and *the interpretation of the last term is based on the assumption that w at a corner has the same value for all domains of regularity meeting at this corner.*

An alternative variational principle involving three fields uses a constrained moment field that corresponds to the given load field, and free and independent deflection and curvature fields. When the functional (11) is transformed by the use of the divergence theorem, the equation of equilibrium, and the boundary conditions on M and V, the following functional is obtained

$$I_2(m, w, \varkappa) = \int \{C_{ij}\varkappa_{ij}\varkappa_{kl} - m_{ij}\varkappa_{ij}\}\,\mathrm{d}A - \int_\theta M\bar{\theta}\,\mathrm{d}s + \int_w V\bar{w}\,\mathrm{d}s - \int_M M\theta\,\mathrm{d}\sigma + \int_V Vw\,\mathrm{d}\sigma - [Tw] \quad \text{(constrained moments corresponding to } p). \tag{13}$$

The natural fields of moment, deflection, and curvature are characterized by the variational equation $\delta I_2 = 0$.

4. TWO INDEPENDENT FIELDS

To derive variational principles with two independent fields from the principles of the preceding section, assume that two of the fields in the latter principles correspond to each other. If, for instance, the curvature field in the functional (11) is assumed to correspond to the moment field, and if this correspondence is used to eliminate the curvature field, the functional is reduced to

$$J_1(m, w) = -\int \{m_{ij}(\partial_{ij}w + \tfrac{1}{2}\bar{C}_{ijkl}m_{kl}) + pw\}\,\mathrm{d}A + \int_M \bar{M}\theta\,\mathrm{d}s - \int_V \bar{V}w\,\mathrm{d}s - \int_\theta M(\bar{\theta} - \theta)\,\mathrm{d}s + \int_w V(\bar{w} - w)\,\mathrm{d}s + \int_\theta M\theta\,\mathrm{d}\sigma - \int_w Vw\,\mathrm{d}\sigma. \tag{14}$$

The variational equation $\delta J_1 = 0$ then furnishes the fundamental relations in (12) except for Hooke's law, which has been stipulated.

Again restricting the discussion to constrained moment fields that correspond to the given load field, one may transform the functional (14) to obtain

$$J_2(m, w) = -\int \tfrac{1}{2}\bar{C}_{ijkl}m_{ij}m_{kl}\,\mathrm{d}A - \int_\theta M\bar{\theta}\,\mathrm{d}s + \int_w V\bar{w}\,\mathrm{d}s - \int_M M\theta\,\mathrm{d}\sigma + \int_V Vw\,\mathrm{d}\sigma - [Tw] \quad \text{(constrained moments corresponding to } p), \tag{15}$$

with $\delta J_2 = 0$ characterizing the natural moments and deflections. In the absence of corners on the boundary, and jumps $\langle M \rangle$ and $\langle V \rangle$ in the interior of the plate, the functional J_2 reduces to the negative complementary energy which involves only the moment field.

Another way of deriving a variational principle with two independent fields is to assume that the curvatures in (11) correspond to the deflections: $\varkappa_{ij} = -\partial_{ij}w$. The functional (11) then simplifies to

$$J_3(m, w) = \int \{\tfrac{1}{2}C_{ijkl}\,\partial_{ij}w\,\partial_{kl}w - pw\}\,\mathrm{d}A + \int_M \bar{M}\theta\,\mathrm{d}s - \int_V \bar{V}w\,\mathrm{d}s - \int_\theta M(\bar{\theta} - \theta)\,\mathrm{d}s + \int_w V(\bar{w} - w)\,\mathrm{d}s + \int_\theta M\theta\,\mathrm{d}\sigma - \int_w Vw\,\mathrm{d}\sigma. \tag{16}$$

The natural moments and deflections are then characterized by $\delta J_3 = 0$. When w and θ satisfy the boundary conditions and $\langle w \rangle = \langle \theta \rangle = 0$, the last four integrals in (16) may be deleted and J_3 reduces to the potential energy, which involves only the deflection field.

5. A SINGLE FIELD

Finally, let the curvatures correspond to both deflections and moments. Twice the functional $I_1(m, w, \varkappa)$ in (11) may then be transformed into

$$K(w) = -\int (\partial_{ij}m_{ij}+2p)w\,\mathrm{d}A - \int_M (M-2\bar{M})\theta\,\mathrm{d}s + \int_V (V-2\bar{V})w\,\mathrm{d}s - \int_\theta M(2\bar{\theta}-\theta)\,\mathrm{d}s$$
$$+\int_w V(2\bar{w}-w)\,\mathrm{d}s - \int_M M\theta\,\mathrm{d}\sigma + \int_\theta M\theta\,\mathrm{d}\sigma + \int_V Vw\,\mathrm{d}\sigma - \int_w Vw\,\mathrm{d}\sigma - [Tw], \tag{17}$$

where

$$m_{ij} = C_{ijkl}\partial_{kl}\, w \tag{18}$$

and M and V are obtained from m_{ij} in accordance with (6). The natural deflections are characterized by $\delta K = 0$.

6. APPLICATIONS

If the plate is covered by a grid with triangular meshes, a continuous, meshwise linear deflection field is completely specified by the deflections at the nodes of the grid. While the deflection (17), and hence the slope along the common side of adjacent meshes is continuous across this side, the slope in the direction normal to the side is not in general continuous. Since $\langle\theta\rangle \neq 0$, the first equation (9) requires that $\langle M\rangle = 0$. Two ways of satisfying this continuity condition will be considered.

(α) A *constant* moment field is specified for each triangular mesh by the values of the normal moment M on the three sides. This yields $\langle M\rangle = 0$ and $\langle V\rangle = 0$ but not, in general, $\langle T\rangle = 0$.

(β) A linear moment field is specified for each triangular mesh by the values of m_{11}, m_{22}, and m_{12} at the three corners. This yields $\langle M\rangle = 0$ and $\langle T\rangle = 0$, but not, in general, $\langle V\rangle = 0$.

For the first choice, $\partial_{ij}w = 0$ and $V = 0$. The functional (14) therefore reduces to

$$J_1(m, w) = -\int \{\tfrac{1}{2}\bar{C}_{ijkl}m_{ij}m_{kl}+pw\}\,\mathrm{d}A + \int_M \bar{M}\theta\,\mathrm{d}s - \int_V \bar{V}w\,\mathrm{d}s$$
$$-\int_\theta M(\bar{\theta}-\theta)\,\mathrm{d}s + \int_\theta M\theta\,\mathrm{d}\sigma. \tag{19}$$

This may be transformed by subtracting (10) in which the two area integrals vanish. Thus,

$$J_1(m, w) = -\int \{\tfrac{1}{2}\bar{C}_{ijkl}m_{ij}m_{kl}+pw\}\,\mathrm{d}A + \int_M (\bar{M}-M)\theta\,\mathrm{d}s$$
$$-\int_V \bar{V}w\,\mathrm{d}s - \int_\theta M\bar{\theta}\,\mathrm{d}s - [Tw]. \tag{20}$$

The last term in (20) may be written as

$$-[Tw] = -\int \frac{\mathrm{d}}{\mathrm{d}\sigma}(Tw)\,\mathrm{d}\sigma = -\int T\frac{\partial w}{\partial \sigma}\,\mathrm{d}\sigma \tag{21}$$

because T, which is derived from a constant moment field in the mesh is constant along a typical mesh side. When (21) is substituted into (20) a functional is obtained that agrees with the functional used by Herrmann [1] except for the term $\int_M(\bar{M}-M)\theta\,ds$. Now Herrmann demands that the moment field be "one in which the prescribed normal moment boundary conditions are satisfied." Since M is piecewise constant on the boundary while $\bar{M}$ will in general vary continuously, this requirement can only in exceptional cases be fulfilled locally. In all other cases, the requirement must be interpreted as implying that the integral $\int(\bar{M}-M)\,ds$ vanishes for the typical mesh side on those parts of the plate boundary on which M is prescribed. Since θ is constant along such a mesh side, this means that $\int_M(\bar{M}-M)\theta\,ds = 0$. Note that the linear equations derived from the variational principle $\delta J_1 = 0$ involve the values of w at the nodes and the values of M for the sides of the meshes as unknowns.

If the continuous and piecewise linear moment field specified under (β) above is used, the last term in (20) vanishes. Note that the linear equations obtained from $\delta J_1 = 0$ then involve the values of w, m_{11}, m_{22}, and m_{12} at the nodes. While this method may be expected to furnish better results than Herrmann's, particularly as far as moments are concerned, it does require the solution of a substantially larger system of linear equations.

As Pólya [3] pointed out in another context, the use of rectangular meshes with meshwise bilinear fields is often more convenient than that of triangular meshes with meshwise linear fields. If this kind of approximation is used for each of the quantities w, m_{11}, m_{22}, and m_{12}, there are no jumps $\langle M\rangle$ and $\langle T\rangle$ across the mesh sides. While $\partial_{11}w$ and $\partial_{22}w$ vanish identically in the typical mesh, $\partial_{12}w$ has a constant value. Accordingly, the term $-\int m_{12}w\,dA$ must be added on the right of (20), but the last term in this equation may be deleted as before.

Bogner *et al.* [4] and Schaefer [5] have pointed out that $\langle\theta\rangle$ can be made to vanish by the use of bicubic deflection fields in the rectangular meshes. If such a meshwise bicubic deflection field w is used in conjunction with the moment field that corresponds to the curvatures $\varkappa_{ij} = -\partial_{ij}w$, the stationary character of the functional (17) may be used to derive the linear equations for the parameters specifying the deflection field.

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Абстракт—Классические вариационные принципы теории упругих пластинок диктуют более точные условия для полей момента и перемещения, чем зто вытекает из сравнения с естественными полями. Так как эти условия непреривности очень трудны для выполнения при подходе для конечного элемента, поэтому нельзя использовать классических вариационных принципов, для определения основных уравнений для типичного метода конечного элемента. Предлагаются модифицированные вариационные принципы, которые допускают менее точные условия непреривности для относительных полей. Эти принципы классификуются в согласии с числом независимых, вызванных полей. Указано применение принципов для расчета типичного конечного элемента.